

# Operators on a non locally compact group algebra.

S.V. Ludkovsky.

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## Abstract

The article is devoted to the investigation of operators on a non locally compact group algebra. Their isomorphisms are also studied.

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## 1 Introduction.

Group algebras play very important role in algebra, harmonic analysis and operator theory [6, 7, 8, 9, 10, 15]. Group algebras were extensively studied for locally compact groups. One of the main instruments in those investigations was an existence of a Haar measure, which is characterized by such essential properties as of being left or right invariant and quasi-invariant relative left and right shifts and to the inversion on the entire group.

But substantially less is known for non locally compact groups. If a nontrivial Borel measure on a topological Hausdorff group quasi-invariant relative to the entire group is given, then such group is locally compact according to A. Weil's theorem. Therefore, on non locally compact Hausdorff groups Borel measures may be quasi-invariant relative to proper subgroups

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 address: Department of Applied Mathematics,  
 Moscow State Technical University MIREA,  
 av. Vernadsky 78, Moscow 119454, Russia  
 Ludkowski@mirea.ru

only. This is the reason of many differences between group algebras of locally compact and non locally compact groups. For non locally compact groups they are already nonassociative. This work continues previous publications of the author.

In this article families of topological groups which may be non locally compact are considered. Group algebras of non locally compact Hausdorff topological groups are studied. Particularly, operators on non locally compact group algebras and their isomorphisms are investigated. Borel regular radonian measures  $\mu_\alpha$  on topological groups  $G_\alpha$  quasi-invariant relative to dense subgroups  $G_\beta$  are taken. The Radon and Borel regularity properties for measures are not very restrictive (see chapter 1 in [2] and chapter 2 in [5]). The constructions of such measures were described in [2, 3, 11, 12, 13, 14] and references therein.

The main results of this paper are obtained for the first time and are contained in Theorems 11, 15, 16, 18.

## 2 Group algebra

**1. Definition.** Let  $\Lambda$  be a directed set and  $\{G_\alpha : \alpha \in \Lambda\}$  be a family of topological groups with completely regular (i.e.  $T_1 \cap T_{3\frac{1}{2}}$ ) topologies  $\tau_\alpha$  such that

- (1)  $\theta_\alpha^\beta : G_\beta \rightarrow G_\alpha$  is a continuous algebraic embedding with continuous inverse  $(\theta_\alpha^\beta)^{-1}$ ,  $\theta_\alpha^\beta(G_\beta)$  is a proper subgroup in  $G_\alpha$  for each  $\alpha < \beta \in \Lambda$ ;
- (2)  $\tau_\alpha \cap \theta_\alpha^\beta(G_\beta) \subset \theta_\alpha^\beta(\tau_\beta)$  and  $\theta_\alpha^\beta(G_\beta)$  is dense in  $G_\alpha$  for each  $\alpha < \beta \in \Lambda$ ;
- (3)  $G_\alpha$  is complete relative to the left uniformity with entourages of the diagonal of the form  $\mathcal{U} = \{(h, g) : h, g \in G_\alpha; h^{-1}g \in U\}$  with neighborhoods  $U$  of the unit element  $e_\alpha$  in  $G_\alpha$ ,  $U \in \tau_\alpha$ ,  $e_\alpha \in U$ ;

(4) for each  $\beta = \phi(\alpha)$  the embedding  $\theta_\alpha^\beta$  is precompact, that is by our definition for every open set  $U$  in  $G_\beta$  containing the unit element  $e_\beta$  a neighborhood  $V \in \tau_\beta$  of  $e_\beta$  exists so that  $V \subset U$  and  $\theta_\alpha^\beta(V)$  is precompact in  $G_\alpha$ , i.e. its closure  $cl(\theta_\alpha^\beta(V))$  in  $G_\alpha$  is compact, where  $\phi : \Lambda \rightarrow \Lambda$  is an increasing marked mapping.

**2. Definition.** Suppose that

- (1)  $\mu_\alpha : \mathcal{B}(G_\alpha) \rightarrow [0, 1]$  is a probability measure on the Borel  $\sigma$ -algebra  $\mathcal{B}(G_\alpha)$  of a group  $G_\alpha$  from §1 with  $\mu_\alpha(G_\alpha) = 1$  so that
- (2)  $\mu_\alpha$  is quasi-invariant relative to the left and right shifts on  $h \in \theta_\alpha^\beta(G_\beta)$

for each  $\alpha < \beta \in \Lambda$ , where  $\rho_{\mu_\alpha}^r(h, g) = (\mu_\alpha^h)(dg)/\mu(dg)$  and  $\rho_{\mu_\alpha}^l(h, g) = (\mu_{\alpha_h})(dg)/\mu(dg)$  denote quasi-invariance  $\mu_\alpha$ -integrable factors,  $\mu_\alpha^h(S) = \mu(Sh^{-1})$  and  $\mu_{\alpha,h}(S) = \mu_\alpha(h^{-1}S)$  for each Borel subset  $S$  in  $G_\alpha$ . Moreover,

(3) let a density  $\psi_\alpha(g) = \mu_\alpha(dg^{-1})/\mu_\alpha(dg)$  relative to the inversion exist and let it be  $\mu_\alpha$ -integrable.

A subset  $E$  in  $G_\alpha$  has  $\mu_\alpha$ -measure zero, if a Borel subset  $F$  in  $G_\alpha$  exists such that  $E \subset F$  and  $\mu_\alpha(F) = 0$ . The completion of  $\mathcal{B}(G_\alpha)$  by all  $\mu_\alpha$ -zero sets will be denoted by  $\mathcal{A}(G_\alpha)$ . The measure  $\mu_\alpha$  has the extension  $\nu_\alpha : 2^{G_\alpha} \rightarrow [0, 1]$  such that  $\nu_\alpha(E) := \inf\{\mu_\alpha(F) : E \subset F \text{ and } F \in \mathcal{B}(G_\alpha)\}$ , where  $2^{G_\alpha}$  denotes the family of all subsets in  $G_\alpha$ . The measure  $\nu_\alpha$  is Borel regular, that is, by the definition all open subsets in  $G_\alpha$  are  $\nu_\alpha$ -measurable and each subset  $E$  in  $G_\alpha$  is contained in a Borel subset  $F$  so that  $\nu_\alpha(E) = \nu_\alpha(F)$ . Evidently,  $\nu_\alpha(F) = \mu_\alpha(F)$  for each Borel subset  $F$  in  $G_\alpha$ , so  $\nu_\alpha$  on  $2^{G_\alpha}$  will also be denoted by  $\mu_\alpha$ .

Henceforth, it will be supposed that

(4) a subset  $W_\alpha \in \mathcal{A}(G_\alpha)$  exists such that  $\rho_{\mu_\alpha}^r(h, g)$  and  $\rho_{\mu_\alpha}^l(h, g)$  are continuous on  $\theta_\alpha^\beta(G_\beta) \times W_\alpha$  and  $\psi_\alpha(g)$  is continuous on  $W_\alpha$  with  $\mu_\alpha(W_\alpha) = 1$  for each  $\alpha \in \Lambda$  with  $\beta = \phi(\alpha)$ . Let also each measure  $\mu_\alpha$  be radonian, that is for each  $\epsilon > 0$  a compact subset  $V$  in  $G_\alpha$  exists such that  $\mu_\alpha(G_\alpha \setminus V) < \epsilon$ .

**3. Notation.** Denote by  $L_{G_\beta}^1(G_\alpha, \mu_\alpha)$  a complex subspace in  $L^1(G_\alpha, \mu_\alpha, \mathbf{C})$ , which is the completion of the linear space  $L^0(G_\alpha, \mathbf{C})$  of all simple functions

$$f(x) = \sum_{j=1}^n b_j \chi_{F_j}(x),$$

where  $b_j \in \mathbf{C}$ ,  $F_j \in \mathcal{A}(G_\alpha)$ ,  $F_j \cap F_k = \emptyset$  for each  $j \neq k$ ,  $\chi_F$  denotes the characteristic function of a subset  $F$ ,  $\chi_F(x) = 1$  for each  $x \in F$  and  $\chi_F(x) = 0$  for every  $x \in G_\alpha \setminus F$ ,  $n \in \mathbf{N}$ . A norm on  $L_{G_\beta}^1(G_\alpha)$  is by our definition given by the formula:

$$(1) \quad \|f\|_{L_{G_\beta}^1(G_\alpha)} := \sup_{h \in \theta_\alpha^\beta(G_\beta)} \|f_h\|_{L^1(G_\alpha)} < \infty,$$

where  $f_h(g) := f(h^{-1}g)$  for  $h, g \in G_\alpha$ ,  $L^1(G_\alpha, \mu_\alpha, \mathbf{C})$  is the usual Banach space of all  $\mu_\alpha$ -measurable functions  $u : G_\alpha \rightarrow \mathbf{C}$  such that

$$(2) \quad \|u\|_{L^1(G_\alpha)} = \int_{G_\alpha} |u(g)| \mu_\alpha(dg) < \infty.$$

Suppose that

(3)  $\phi : \Lambda \rightarrow \Lambda$  is an increasing mapping,  $\alpha < \phi(\alpha)$  for each  $\alpha \in \Lambda$ . We consider the complex space

(4)  $L^\infty(L_{G_\beta}^1(G_\alpha, \mu_\alpha) : \alpha < \beta \in \Lambda) := \{f = (f_\alpha : \alpha \in \Lambda); f_\alpha \in L_{G_\beta}^1(G_\alpha, \mu_\alpha) \text{ for each } \alpha \in \Lambda; \|f\|_\infty := \sup_{\alpha \in \Lambda} \|f_\alpha\|_{L_{G_\beta}^1(G_\alpha)} < \infty, \text{ where } \beta = \phi(\alpha)\}$ .

When measures  $\mu_\alpha$  are specified, spaces are denoted shortly by  $L_{G_\beta}^1(G_\alpha)$  and  $L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$ .

**4. Proposition.** *Supply the family  $L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$  from §3 with the multiplication  $f \tilde{*} u = w$  such that*

$$(1) \quad w_\alpha = f_\beta \tilde{*} u_\alpha = \int_{G_\beta} f_\beta(h) u_\alpha(\theta_\alpha^\beta(h)g) \mu_\beta(dh).$$

*Then  $L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$  is the complex normed algebra generally noncommutative and nonassociative.*

**Proof.** Evidently  $L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$  is the complex linear normed space, since

$$\begin{aligned} \|af + bu\|_\infty &:= \sup_{\alpha \in \Lambda} \|af_\alpha + bu_\alpha\|_{L_{G_\beta}^1(G_\alpha)} \leq \sup_{\alpha \in \Lambda} |a| \|f_\alpha\|_{L_{G_\beta}^1(G_\alpha)} + \sup_{\alpha \in \Lambda} |b| \|u_\alpha\|_{L_{G_\beta}^1(G_\alpha)} \\ &= |a| \|f\|_\infty + |b| \|u\|_\infty \end{aligned}$$

for each functions  $f, u \in L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$  and complex numbers  $a, b \in \mathbf{C}$ , where  $\beta = \phi(\alpha)$  for each  $\alpha \in \Lambda$ . On the other hand,

$$\|f_\beta \tilde{*} u_\alpha\|_{L_{G_\beta}^1(G_\alpha)} \leq \|f_\beta\|_{L^1(G_\beta)} \|u_\alpha\|_{L_{G_\beta}^1(G_\alpha)} \leq \|f_\beta\|_{L_{G_\beta}^1(G_\alpha)} \|u_\alpha\|_{L_{G_\beta}^1(G_\alpha)}$$

in accordance with Lemma 17.2 [12], since  $\|f_\beta\|_{L^1(G_\beta)} \leq \|f_\beta\|_{L_{G_\beta}^1(G_\alpha)}$ , consequently,

$$\|f \tilde{*} u\|_\infty \leq \|f\|_\infty \|u\|_\infty.$$

The noncommutativity and the nonassociativity of this multiplication follows from Formula (1).

**5. Corollary.** *The family of all nonnegative functions  $\mathcal{P} := \{f : f \in L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda); f_\alpha(x) \geq 0 \forall x \in G_\alpha \forall \alpha \in \Lambda\}$  is a lattice.*

**Proof.** If  $f, u \in \mathcal{P}$  and  $a \geq 0$  and  $b \geq 0$ , then  $af + bu \in \mathcal{P}$ ,  $f \wedge u = \min(f, u) \in \mathcal{P}$ ,  $f \vee g = \max(f, g) \in \mathcal{P}$ , since  $af_\alpha(x) + bu_\alpha(x) \geq 0$  and

$\min(f_\alpha(x), u_\alpha(x)) \geq 0$  and  $\max(f_\alpha(x), u_\alpha(x)) \geq 0$  for each  $x \in G_\alpha$  and  $\alpha \in \Lambda$ . Moreover, from  $\mu_\alpha(S) \geq 0$  for each  $S \in \mathcal{B}(G_\alpha)$  and  $\alpha \in \Lambda$  and from Formula 4(1) it follows that  $(f \tilde{*} u)_\alpha(x) \geq 0$  for each  $x \in G_\alpha$  and  $\alpha \in \Lambda$ , consequently,  $f \tilde{*} u \in \mathcal{P}$  for all  $f, u \in \mathcal{P}$ .

**6. Corollary.** *The operators  $T_f$  and  $\tilde{T}_f$  defined by the formulas*

(1)  $T_f u := f \tilde{*} u$  and

(2)  $\tilde{T}_f u := u \tilde{*} f$  are  $\mathbf{C}$ -linear and continuous on the algebra  $L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$  for each  $f \in L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$ .

**7. Lemma.** *Let an operator  $\hat{U}_g$  be given by the formula:*

(1)  $\hat{U}_{g_\beta} f_\alpha(x) = f_\alpha(\theta_\alpha^\beta(g_\beta)x)$  for each  $\alpha \in \Lambda$  with  $\beta = \phi(\alpha)$ ,  $g_\beta \in G_\beta$ ,  $x \in G_\alpha$ , then  $\hat{U}_g : L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda) \rightarrow L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$  is the linear isometry for each  $g = \{g_\beta : \beta \in \Lambda, g_\beta \in G_\beta\}$ .

**Proof.** Evidently,  $\hat{U}_g(af + bu) = a\hat{U}_g f + b\hat{U}_g u$  for each  $a, b \in \mathbf{C}$  and  $f, u \in L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$ , since  $\hat{U}_{g_\beta}(af_\alpha(x) + bu_\alpha(x)) = af_\alpha(\theta_\alpha^\beta(g_\beta)x) + bf_\alpha(\theta_\alpha^\beta(g_\beta)x) = a\hat{U}_{g_\beta} f_\alpha(x) + b\hat{U}_{g_\beta} u_\alpha(x)$  for each  $\alpha \in \Lambda$  with  $\beta = \phi(\alpha)$ ,  $g_\beta \in G_\beta$ ,  $x \in G_\alpha$ . The isometry property follows from the equalities:

$$\begin{aligned} \|\hat{U}_g f\|_\infty &= \sup_{\alpha \in \Lambda, \beta = \phi(\alpha)} \sup_{h \in \theta_\alpha^\beta(G_\beta)} \int_{G_\alpha} |f_\alpha(h\theta_\alpha^\beta(g_\beta)x)| \mu_\alpha(dx) \\ &= \sup_{\alpha \in \Lambda, \beta = \phi(\alpha)} \sup_{h \in \theta_\alpha^\beta(G_\beta)} \int_{G_\alpha} |f_\alpha(hx)| \mu_\alpha(dx) = \|f\|_\infty. \end{aligned}$$

**8. Lemma.** *The operator  $\hat{U}_g$  from §7 satisfies the equality:*

$$(1) \quad \hat{U}_g(f \tilde{*} u) = f \tilde{*} \hat{U}_g u$$

for each  $f, u \in L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$  and every  $g = \{g_\beta : \beta \in \Lambda, g_\beta \in G_\beta\}$ .

**Proof.** Formula (1) follows from the equalities:

$$\hat{U}_{g_\beta}(f_\beta \tilde{*} u_\alpha)(x) = \int_{G_\beta} f_\beta(h) u_\alpha((\theta_\alpha^\beta(hg_\beta)x)) \mu_\beta(dh) = (f_\beta \tilde{*} (\hat{U}_{g_\beta} u_\alpha))(x)$$

for each  $x \in G_\alpha$ ,  $\alpha \in \Lambda$ ,  $\beta = \phi(\alpha)$ .

**9. Corollary.** *A bijective correspondence between elements  $g \in \prod_{\alpha \in \Lambda} G_\alpha =: G$  and operators  $\hat{U}_g$  exists so that*

(1)  $\hat{U}_g \hat{U}_h = \hat{U}_{gh}$  for each  $g, h \in G$ .

**Proof.** The group  $G$  consists of elements  $g = \{g_\alpha : \alpha \in \Lambda, g_\alpha \in G_\alpha\}$  with the multiplication  $gh = \{g_\alpha h_\alpha : \alpha \in \Lambda\}$  and the inversion  $g^{-1} = \{g_\alpha^{-1} : \alpha \in \Lambda\}$ , since  $g_\alpha h_\alpha \in G_\alpha$  for each  $\alpha \in \Lambda$ . This group  $G$  is the topological group relative to the Tychonoff (product) topology  $\tau^t$  with the base  $V = V_\alpha \times \prod_{\gamma \in \Lambda; \gamma \neq \alpha} G_\gamma$ , where  $V_\alpha \in \tau_\alpha$ ,  $\tau_\alpha$  is the topology on  $G_\alpha$ ,  $\alpha \in \Lambda$ . It is also the topological group relative to the box topology  $\tau^b$  with the base  $V = \prod_{\alpha \in \Lambda} V_\alpha$ , where  $V_\alpha \in \tau_\alpha$  is open in  $G_\alpha$  for each  $\alpha \in \Lambda$ . Then we deduce that

$$\begin{aligned} \hat{U}_{g_\beta} \hat{U}_{h_\beta} f_\alpha(x) &= \hat{U}_{g_\beta}(\hat{U}_{h_\beta} f_\alpha(x)) = \hat{U}_{g_\beta}(f_\alpha(\theta_\alpha^\beta(h_\beta)x)) = \\ f_\alpha(\theta_\alpha^\beta(g_\beta)(\theta_\alpha^\beta(h_\beta)x)) &= f_\alpha(\theta_\alpha^\beta(g_\beta h_\beta)x) = \hat{U}_{g_\beta h_\beta} f_\alpha(x) \end{aligned}$$

for each  $\alpha \in \Lambda$  with  $\beta = \phi(\alpha)$ , where  $x \in G_\alpha$ . The latter relation implies Formula (1).

**10. Proposition.** *The representation  $\hat{U} : (G, \tau^b) \rightarrow L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$  is strongly continuous.*

**Proof.** Each bounded either continuous or simple function  $f : G_\alpha \rightarrow \mathbf{C}$  evidently belongs to  $L_{G_\beta}^1(G_\alpha)$ , since  $\mu_\alpha$  is the probability measure and

$$(1) \quad \|f\|_{L_{G_\beta}^1(G_\alpha)} \leq \sup_{x \in G_\alpha} |f(x)| < \infty.$$

Each compact subset  $V$  in  $G_\alpha$  is closed in  $G_\alpha$  in accordance with Theorem 3.1.8 [4], consequently, every compact subset  $V$  is a Borel subset.

For an arbitrary marked function  $u \in L_{G_\beta}^1(G_\alpha)$  from the inclusion  $L_{G_\beta}^1(G_\alpha) \subset L^1(G_\alpha)$ , the Borel regularity of the measure  $\mu_\alpha$ , Conditions 2(1–4) and Lusin's theorem 2.3.5 [5] it follows that for each  $\epsilon_n > 0$  a compact subset  $E_n = E_n(u)$  in  $G_\alpha$  exists so that the restriction  $u|_{E_n}$  is continuous and  $\mu_\alpha(G_\alpha \setminus E_n) < \epsilon_n$ , since the measure  $\mu_\alpha$  is radonian and for each  $\delta > 0$  a compact subset  $V$  in  $G_\alpha$  exists such that  $\mu_\alpha(G_\alpha \setminus V) < \delta$  and considering  $u|_V$  and  $\mu_\alpha|_{\mathcal{B}(V)}$ . Take a monotone decreasing sequence  $\epsilon_n$  such that  $\epsilon_n \downarrow 0$ . Then  $\mu_\alpha(G_\alpha \setminus E) = 0$ , where

$$E(u) = E := \bigcup_{n=1}^{\infty} E_n.$$

For every  $\delta > 0$  and each restriction  $u|_{E_n}$  a simple function

$$v_n = \sum_{j=1}^{m(n)} b_{j,n} \chi_{F_{j,n}}(x)$$

exists such that

$$\sup_{x \in E_n} |u(x) - v_n(x)| < \delta,$$

where  $b_{j,n} \in \mathbf{C}$ ,  $F_{j,n} \in \mathcal{B}(E_n)$ ,  $m(n) \in \mathbf{N}$ . Put  $v_0(x) = 0$  on  $G_\alpha \setminus E$  and take the combination  $v$  of these mappings  $v_n$ , then  $\sup_{x \in E} |u(x) - v(x)| < \delta$  and hence  $v \in L_{G_\beta}^1(G_\alpha)$  due to Inequality (1).

In accordance with §3 in each space  $L_{G_\beta}^1(G_\alpha)$  the linear space of all simple functions

$$(2) \quad f(x) = \sum_{j=1}^p b_j \chi_{F_j}(x)$$

is dense, where  $b_j \in \mathbf{C}$ ,  $F_j \in \mathcal{A}(G_\alpha)$ ,  $\chi_F$  denotes the characteristic function of a subset  $F$ ,  $\chi_F(x) = 1$  for each  $x \in F$  and  $\chi_F(x) = 0$  for every  $x \in G_\alpha \setminus F$ ,  $p \in \mathbf{N}$ .

In view of Lemma 7 it is sufficient to prove, that the representation  $G \ni g \mapsto \hat{U}_g f_\alpha$  is continuous on each simple function  $f_\alpha \in L_{G_\beta}^1(G_\alpha)$ , when  $G$  is supplied with the box topology  $\tau^b$ , since  $\|\hat{U}_g(f_\alpha - u)\| = \|f_\alpha - u\|$ .

Now we take  $E_n = E_n(f_\alpha)$  and  $E = E(f_\alpha)$  as above. The measure  $\mu_\alpha$  is quasi-invariant, consequently,

(3)  $\mu_\alpha(h^{-1}(G_\alpha \setminus E)) = 0$  and hence  $\mu_\alpha([h^{-1}E] \cap W_\alpha) = 1$  for each  $h \in \theta_\alpha^\beta(G_\beta)$  with  $\beta = \phi(\alpha)$  and  $\alpha \in \Lambda$ .

On the other hand,

(4)  $\hat{U}_{g_\beta} f_\alpha(x) - \hat{U}_{e_\beta} f_\alpha(x) = f_\alpha(\theta_\alpha^\beta(g_\beta)x) - f_\alpha(x)$  and

$$\|[\hat{U}_{g_\beta} - \hat{U}_{e_\beta}] f_\alpha(x)\|_{L_{G_\beta}^1(G_\alpha)} = \sup_{h \in \theta_\alpha^\beta(G_\beta)} \int_{G_\alpha} |f(h\theta_\alpha^\beta(g_\beta)x) - f(hx)| \mu_\alpha(dx) \leq 2\|f\|_{L_{G_\beta}^1(G_\alpha)} < \infty$$

for each  $\alpha \in \Lambda$  with  $\beta = \phi(\alpha)$ ,  $g_\beta \in G_\beta$ ,  $x \in G_\alpha$ , since  $\theta_\alpha^\beta(e_\beta) = e_\alpha$  is the unit element in the group  $G_\alpha$ .

For each  $\delta > 0$  an element  $h_\delta \in \theta_\alpha^\beta(G_\beta)$  exists such that

$$(5) \quad \left| \|f_\alpha(x)\|_{L_{G_\beta}^1(G_\alpha)} - \int_{G_\alpha} |f_\alpha(h_\delta x)| \mu_\alpha(dx) \right| < \delta.$$

Evidently, the series

$$(6) \quad \int_{G_\alpha} |f(hx)| \mu_\alpha(dx) = \sum_{j=1}^p |b_j| \mu_\alpha(h^{-1}F_j) \leq \sum_{j=1}^p |b_j|$$

is finite, since  $\mu_\alpha$  is the nonnegative measure and  $\mu_\alpha(G_\alpha) = 1$  and  $p \in \mathbf{N}$  is a natural number. Each measure

$$(7) \quad w_h(A) := \sum_{j=1}^p |b_j| \mu_\alpha((h^{-1}F_j) \cap A)$$

is  $\sigma$ -additive on  $\mathcal{A}(G_\alpha)$  and absolutely continuous relative to  $\mu_\alpha$  due Formula (6), where  $A \in \mathcal{A}(G_\alpha)$ ,  $h \in G_\beta$ .

For a given arbitrary positive number  $\epsilon > 0$  take a natural number  $n_0$  such that  $\epsilon > \epsilon_n$  for each  $n > n_0$ . Choose a marked natural number  $m > n_0$ . From Theorem 4.5 [8] and Formulas (1 – 4, 7) and Conditions 1(1 – 4) and 2(1 – 4) it follows that for each  $\delta > 0$  a symmetric  $V_\beta = V_\beta^{-1}$  neighborhood of  $e_\beta$  in  $G_\beta$  exists such that  $\theta_\alpha^\beta(V_\beta)$  is precompact in  $G_\alpha$  and

$$(8) \quad \int_{G_\alpha \cap E_m} \left| \sum_{j=1}^p b_j \chi_{F_j}(\theta_\alpha^\beta(g_\beta)hx) - \sum_{j=1}^p b_j \chi_{F_j}(hx) \right| \mu_\alpha(dx) < \delta.$$

This is possible by a choice of a sufficiently small  $V_\beta$  such that the left quasi-invariance factor  $\rho^l(a, b)$  is bounded on  $V_\beta \times (W_\alpha \cap E_m)$ , since  $\rho_l$  is continuous on  $G_\beta \times W_\alpha$  and  $E_m = E_m(f_\alpha)$  is compact. Indeed, the product  $cl(\theta_\alpha^\beta(V_\beta))E_m =: Q_m$  is compact in  $G_\alpha$  as the product of two compact subsets in the topological group  $G_\alpha$  (see §4.4 in [8]) and  $E_m \subset Q_m$ . From the choice of  $E_m$  we infer, that

$$(9) \quad \int_{G_\alpha \setminus E_m} \left| \sum_{j=1}^p b_j \chi_{F_j}(\theta_\alpha^\beta(g_\beta)hx) - \sum_{j=1}^p b_j \chi_{F_j}(hx) \right| \mu_\alpha(dx) \leq 2\epsilon \sum_{j=1}^p |b_j|$$

for each  $h \in \theta_\alpha^\beta(G_\beta)$ .

Thus from (8, 9) it follows, that for each  $\delta > 0$  a neighborhood  $V_\beta$  of  $e_\beta$  in  $G_\beta$  exists such that  $\|[\hat{U}_{g_\beta} - \hat{U}_{e_\beta}]f_\alpha(x)\|_{L_{G_\beta}^1(G_\alpha)} < \delta$  for each  $g_\beta \in V_\beta$ . Taking  $V = \prod_{\alpha \in \Lambda} V_\alpha$  we get that  $\|[\hat{U}_g - \hat{U}_e]f\| < \delta$  for every  $g \in V$ , where  $f = (f_\alpha : \alpha \in \Lambda) \in L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$ .

**11. Theorem.** *Suppose that a continuous mapping  $S : L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda) \rightarrow L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$  satisfies the following conditions:*

- (1)  *$S$  is linear over the complex field so that  $Sf = (S_\alpha f_\alpha : \alpha \in \Lambda)$  with  $S_\alpha f_\alpha \in L_{G_\beta}^1(G_\alpha)$  for each  $\alpha \in \Lambda$ , where  $\beta = \phi(\alpha)$ ;*
- (2) *positive, i.e.  $S_\alpha f_\alpha$  is positive if  $f_\alpha$  is positive for each  $\alpha \in \Lambda$ ;*



(3)  $S(f\tilde{\star}g) = f\tilde{\star}Sg$  for every  $f, g \in L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$ .

Then elements  $a \in G$  and  $p = \{p_\alpha : p_\alpha > 0 \ \forall \alpha \in \Lambda\} \in \mathbf{R}^\Lambda$  exist so that

(4)  $S = p\hat{U}_a$ , that is  $S_\alpha f_\alpha(x) = p_\alpha \hat{U}_{a_\beta} f_\alpha(x)$  for any  $\alpha \in \Lambda$  with  $\beta = \phi(\alpha)$  and each  $x \in G_\alpha$ .

**Proof.** In view of Corollary 5 the family  $\mathcal{P}$  of all nonnegative elements forms the cone. Conditions (1, 2) imply that

(5)  $S(f \vee g) = (Sf) \vee (Sg)$  and  $S(f \wedge g) = (Sf) \wedge (Sg)$  and

(6)  $S(\bigvee_{n=1}^\infty g_n) = \bigvee_{n=1}^\infty (Sg_n)$  and  $S(\bigwedge_{n=1}^\infty g_n) = \bigwedge_{n=1}^\infty (Sg_n)$  on  $\mathcal{P}$ .

Being continuous the operator  $S$  is bounded. We consider a subset  $E = \prod_{\alpha \in \Lambda} E_\alpha$  such that  $E_\alpha \in \mathcal{A}(G_\alpha)$  for each  $\alpha$ . The function

(7)  $\xi_E = \{\chi_{E_\alpha} : \alpha \in \Lambda\}$  belongs to  $L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$ , since  $\int_{G_\alpha} \chi_{E_\alpha}(x) \mu_\alpha(dx) = \mu_\alpha(E_\alpha) \leq \mu_\alpha(G_\alpha) = 1$  for any  $\alpha$ . The function  $\eta_E(x) := S\xi_E(x)$  is positive by the conditions of this theorem, hence  $\eta_{E,\alpha} := S_\alpha \chi_{E_\alpha}(x_\alpha) \geq 0$  on  $G_\alpha$  and

(8)  $\eta_{E,\alpha}(x_\alpha) > 0$  for each  $x_\alpha \in T_{S,\alpha}(E_\alpha)$  and every  $\alpha \in \Lambda$ , where  $T_{S,\alpha}(E_\alpha)$  denotes a subset in  $G_\alpha$  on which a function  $\eta_{E,\alpha}$  is positive which is defined up to a  $\mu_\alpha$ -null set. From Formulas (5) we get that

(9)  $T_{S,\alpha}(E_\alpha \cup F_\alpha) = T_{S,\alpha}(E_\alpha) \cup T_{S,\alpha}(F_\alpha)$  and  $T_{S,\alpha}(E_\alpha \cap F_\alpha) = T_{S,\alpha}(E_\alpha) \cap T_{S,\alpha}(F_\alpha)$  for each  $\alpha \in \Lambda$  and  $E_\alpha, F_\alpha \in \mathcal{A}(G_\alpha)$ . Moreover, the definition of  $T_{S,\alpha}(E_\alpha)$  by Formula (8) implies that

(10)  $(T_S)^{-1} = T_{S^{-1}}$ , that is  $(T_{S,\alpha})^{-1} = T_{S^{-1},\alpha}$  for any  $\alpha \in \Lambda$ .

For a marked  $\alpha \in \Lambda$  and  $\beta = \phi(\alpha)$  we next consider a base of symmetric neighborhoods  $U_{\alpha,v} = U_{\alpha,v}^{-1}$  and  $U_{\beta,v} = U_{\beta,v}^{-1}$  of the unit elements  $e_\alpha$  in  $G_\alpha$  and  $e_\beta$  in  $G_\beta$  satisfying the conditions:

(11)  $U_{\beta,v} \subseteq U_{\alpha,v} \cap G_\beta$  for each  $v \in \Upsilon_{\alpha,\beta}$ , where

(12)  $\Upsilon_{\alpha,\beta}$  is a directed set by inclusion:  $v \leq t$  if and only if  $U_{\alpha,t} \subseteq U_{\alpha,v}$  so that for each  $v \in \Upsilon_{\alpha,\beta}$  there exists  $q(v) \in \Upsilon_{\alpha,\beta}$  with  $q(v) > v$  and  $U_{\alpha,q(v)} \subset U_{\alpha,v}$  and  $U_{\alpha,q(v)} \neq U_{\alpha,v}$ ,  $U_{\alpha,v}^{-1} = \{g^{-1} : g \in U_{\alpha,v}\}$ . Then we put

(13)  $\xi_{\alpha,v} = \chi_{U_{\alpha,v}}$ ,  $\eta_{\alpha,v} = S_\alpha \xi_{\alpha,v}$ ,  $\xi_v := \{\xi_{\alpha,v} : \alpha \in \Lambda\}$  and  $\eta_v := \{\eta_{\alpha,v} : \alpha \in \Lambda\}$  for any  $v \in \Upsilon_{\alpha,\beta}$ . Below the proof of this theorem is continued and is based on the following intermediate lemmas.

**12. Lemma.** Let  $P = T_S^{-1}U$ , that is  $P_{\alpha,v} = T_{S,\alpha}^{-1}(U_{\alpha,v})$  for each  $\alpha \in \Lambda$  and  $v \in \Upsilon_{\alpha,\beta}$ , where  $T_S$  and  $U_{\alpha,v}$  and  $\Upsilon_{\alpha,\beta}$  with  $\beta = \phi(\alpha)$  are as in §11. Then for any  $\alpha \in \Lambda$  and for each  $v \in \Upsilon_{\alpha,\beta}$  elements  $w = w(v) \in \Upsilon_{\alpha,\beta}$  and  $a_{\beta,v} \in G_\beta$  exist such that

(1)  $a_{\beta,v} U_{\beta,v} \supset P_{\beta,w(v)}$  up to a  $\mu_\beta$ -null set.

**Proof.** Suppose the contrary that there exists  $U_{\beta,v}$  so that  $a_\beta U_{\beta,v}$  does not cover  $\mu_\beta$ -almost entirely the set  $P_{\beta,w}$  for any  $w \in \Upsilon_{\alpha,\beta}$  and any  $a_\beta \in G_\beta$ . Since  $U_{\alpha,v}$  is a base of neighborhoods of the unit element in  $G_\alpha$  there exist  $t, s \in \Upsilon_{\alpha,\beta}$  such that  $U_{\beta,t}^3 \subseteq U_{\beta,v}$  and  $U_{\beta,s}^2 \subseteq U_{\beta,v}$ . Take two elements  $a_{\beta,1}, a_{\beta,2} \in G_\beta$  satisfying the conditions:

(2)  $A_\alpha = cl_\alpha \theta_\alpha^\beta([a_{\beta,1} U_{\beta,t}] \cap P_{\beta,s})$  and  $B_\alpha = cl_\alpha \theta_\alpha^\beta([a_{\beta,2} U_{\beta,t}] \cap P_{\beta,s})$  and  $A_\beta = [a_{\beta,1} U_{\beta,t}] \cap P_{\beta,s}$  and  $B_\beta = [a_{\beta,2} U_{\beta,t}] \cap P_{\beta,s}$  with  $\mu_\alpha(A_\alpha) > 0$  and  $\mu_\alpha(B_\alpha) > 0$  and  $\mu_\beta(A_\beta) > 0$  and  $\mu_\beta(B_\beta) > 0$  and

(3)  $A_\beta \cap B_\beta U_{\beta,t} = \emptyset$ , where  $cl_\alpha(A)$  denotes the closure of a subset  $A$  in  $G_\alpha$ . This is possible, since the group  $\theta_\alpha^\beta(G_\beta)$  is dense in  $G_\alpha$  and the quasi-invariant radonian Borel regular measures  $\mu_\alpha$  and  $\mu_\beta$  are positive on each open subset in  $G_\alpha$  and  $G_\beta$  correspondingly.

Then the sets

(4)  $C_\alpha = T_{S,\alpha}^{-1}(A_\alpha)$  and  $D_\alpha = T_{S,\alpha}^{-1}(B_\alpha)$  are  $\mu_\alpha$ -measurable and  $C_\alpha \cup D_\alpha \subset cl_\alpha(U_{\beta,s})$ , also  $C_\beta = T_{S,\beta}^{-1}(A_\beta)$  and  $D_\beta = T_{S,\beta}^{-1}(B_\beta)$  are  $\mu_\beta$ -measurable and are contained in  $U_{\beta,s}$ ,  $C_\beta \cup D_\beta \subset U_{\beta,s}$ . From Conditions 11(10,11) we deduce that

(5)  $C_\alpha^{-1} D_\alpha \subset cl_\alpha(U_{\beta,v})$  and  $C_\beta^{-1} D_\beta \subset U_{\beta,v}$ .

Applying Formula 11(3) we infer the following:

$$\begin{aligned} (6) \quad (\chi_{C_\beta} \tilde{*} \chi_{cl_\alpha(U_{\beta,t})})(x) &= \int_{G_\beta} \chi_{C_\beta}(h) \chi_{cl_\alpha(U_{\beta,t})}(\theta_\alpha^\beta(h)x) \mu_\beta(dh) \\ &= \int_{C_\beta} \chi_{cl_\alpha(U_{\beta,t})}(\theta_\alpha^\beta(h)x) \mu_\beta(dh) = \mu_\beta(C_\beta \cap cl_\alpha(U_{\beta,t}x^{-1})), \end{aligned}$$

consequently,  $(\chi_{C_\beta} \tilde{*} \chi_{cl_\alpha(U_{\beta,t})})(x) = \mu_\beta(C_\beta) > 0$  for each  $x \in D_\alpha$ , since  $C_\alpha^{-1} D_\alpha \subset cl_\alpha(U_{\beta,t})$  and  $C_\alpha D_\alpha^{-1} \subset cl_\alpha(U_{\beta,t}^{-1}) = cl_\alpha(U_{\beta,t})$  and hence

$$(7) \quad (\chi_{C_\beta} \tilde{*} \chi_{cl_\alpha(U_{\beta,t})})(x) \geq \mu_\beta(C_\beta) \chi_{D_\alpha}(x).$$

Then the inequality

$$(8) \quad S(\chi_{C_\beta} \tilde{*} \chi_{cl_\alpha(U_{\beta,t})}) = (\chi_{C_\beta} \tilde{*} S \chi_{cl_\alpha(U_{\beta,t})}) \geq \mu_\beta(C_\beta) S \chi_{D_\alpha}$$

follows from Formulas (7) and 11(3). Applying Conditions (2) one gets  $A_\alpha = T_{S,\alpha}(C_\alpha) = \{x : S \chi_{C_\alpha}(x) > 0\}$  and  $B_\alpha = T_{S,\alpha}(D_\alpha) = \{x : S \chi_{D_\alpha}(x) > 0\}$  and  $A_\beta = T_{S,\beta}(C_\beta) = \{x : S \chi_{C_\beta}(x) > 0\}$  and  $B_\beta = T_{S,\beta}(D_\beta) = \{x : S \chi_{D_\beta}(x) > 0\}$ . On the other hand, from Formulas (3) we deduce the following:

$$(9) \quad S(\chi_{C_\beta} \tilde{*} \chi_{cl_\alpha(U_{\beta,t})})(x) = S \int_{G_\beta \cap U_{\beta,t}x^{-1}} \chi_{C_\beta}(h) \mu_\beta(dh) = 0,$$

since  $A_\beta \cap (B_\beta U_{\beta,t}) = \emptyset$  and hence  $T_{S,\beta}^{-1}(A_\beta) \cap T_{S,\beta}^{-1}(B_\beta U_{\beta,t}) = \emptyset$  so that  $C_\beta \cap (D_\beta P_{\beta,t}) = \emptyset$ . But Formula (9) contradicts (7), that finishes the proof of this lemma.

**13. Lemma.** *Let an operation  $\tilde{*}$  on  $M(G_\beta) \times L_{G_\beta}^1(G_\alpha)$  be defined by the formula*

$$(1) \quad (\nu \tilde{*} u)(g) = \int_{G_\beta} \nu(dh) u(\theta_\alpha^\beta(h)g)$$

for each  $g \in G_\alpha$ , where  $M(G_\beta)$  denotes the set of all finite radon measures  $\nu$  on  $G_\beta$  supplied with the norm. Then the mapping  $\tilde{*} : M(G_\beta) \times L_{G_\beta}^1(G_\alpha) \rightarrow L_{G_\beta}^1(G_\alpha)$  is bilinear and continuous.

**Proof.** From Formula (1) it follows that  $(\nu \tilde{*} (au + bg)) = a(\nu \tilde{*} u) + b(\nu \tilde{*} g)$  and  $(a\nu + b\mu) \tilde{*} u = a(\nu \tilde{*} u) + b(\mu \tilde{*} u)$  for any complex numbers  $a, b \in \mathbf{C}$ , radonian measures  $\nu, \mu \in M(G_\beta)$  and functions  $u, g \in L_{G_\beta}^1(G_\alpha)$ . Remind that the space  $M(G_\beta)$  is supplied with the standard norm:  $\|\nu\| = |\nu|(G_\beta)$ , where  $|\nu| = \nu^+ + \nu^-$  is the variation of  $\nu$  with the standard decomposition  $\nu = \nu^+ - \nu^-$  into the difference of two nonnegative measures  $\nu^+$  and  $\nu^-$ . Then

$$\begin{aligned} & \sup_{s \in \theta_\alpha^\beta(G_\beta)} \int_{G_\alpha} \left| \int_{G_\beta} \nu(dh) u(\theta_\alpha^\beta(h)(sg)) \right| \mu_\alpha(dg) \leq \\ & \sup_{s \in \theta_\alpha^\beta(G_\beta)} \int_{G_\beta} \{ |\nu|(dh) \int_{G_\alpha} |u(\theta_\alpha^\beta(h)(sg))| \mu_\alpha(dg) \} \end{aligned}$$

due to Fubini's theorem, consequently,

$$(2) \quad \|\nu \tilde{*} u\|_{L_{G_\beta}^1(G_\alpha)} \leq \|\nu\| \|u\|_{L_{G_\beta}^1(G_\alpha)}.$$

The latter inequality implies the continuity of such skew convolution  $\tilde{*} : M(G_\beta) \times L_{G_\beta}^1(G_\alpha) \rightarrow L_{G_\beta}^1(G_\alpha)$ .

**Continuation of the proof of Theorem 11.** By transfinite induction and Teichmüller-Tukey's lemma applying Lemma 12 one gets a base of symmetric neighborhoods of the unit elements such that

(14)  $U_{\beta,t}^3 \subseteq U_{\beta,v}$  and  $U_{\beta,s}^2 \subseteq U_{\beta,v}$  for each  $t \in \lambda(\beta, v)$  and  $s \in \nu(\beta, v)$ , where  $\lambda(\beta, v) \subset \nu(\beta, v)$  are cofinal subsets in  $\Upsilon_{\alpha,\beta}$  all elements of which are greater than  $v$  (see also §1.3 [4]); and Conditions 11(11, 12) and 12(1). The inclusion  $P_{\beta,v} \supseteq P_{\beta,q}$  for each  $v < q$  leads to  $a_{\beta,v} U_{\beta,v} \cap a_{\beta,q} U_{\beta,q} \neq \emptyset$ , consequently,

$$(15) \quad a_{\beta,v}^{-1} a_{\beta,q} \in U_{\beta,v} U_{\beta,q}^{-1}$$

and hence  $\{a_{\beta,v} : v\}$  is a fundamental (Cauchy) net in  $G_\beta$ . But  $G_\beta$  is the topological group complete relative to its left uniformly (see §§8.1.17 and 8.3 [4]). Therefore, this net converges  $\lim_v a_{\beta,v} = a_\beta$  in  $G_\beta$ . From (15) the inclusion  $a_\beta \in a_{\beta,v} U_{\beta,v}^2$  follows, consequently,

$$(16) \quad a_\beta U_{\beta,v} \supset a_{\beta,s} U_{\beta,s} \supset P_{\beta,w(s)} \text{ for every } s \in \nu(\beta, v).$$

In view of Proposition 17.7 [12]

$$(17) \quad \lim_s \left\| \frac{\chi_{U_{\beta,w(s)}}}{\mu_\beta(U_{\beta,w(s)})} \tilde{*} f - f \right\|_{L_{G_\beta}^1(G_\alpha)} = 0$$

for each  $f \in L_{G_\beta}^1(G_\alpha)$ , since  $\mu_\beta(U_{\beta,w(s)}) > 0$ , where  $\beta = \phi(\alpha)$ .

The left quasi-invariance factor  $\rho_{\mu_\alpha}^l(\theta_\alpha^\beta(h), g)$  is continuous on  $G_\beta \times W_\alpha$  and satisfies the cocycle condition

$$\rho_{\mu_\alpha}^l(\theta_\alpha^\beta(h), \theta_\alpha^\beta(t^{-1})g) \rho_{\mu_\alpha}^l(\theta_\alpha^\beta(t), g) = \rho_{\mu_\alpha}^l(\theta_\alpha^\beta(ht), g)$$

for all  $h, t \in G_\beta$  and  $g \in G_\alpha$ . The probability measure  $\mu_\alpha$  is Borel regular and  $\mu_\alpha(G_\alpha) = \mu_\alpha(W_\alpha)$ , consequently,  $W_\alpha$  is dense in  $G_\alpha$  and hence has a continuous extension onto  $G_\beta \times G_\alpha$  due to the cocycle condition and since  $\tau_\alpha \cap \theta_\alpha^\beta(G_\beta) \subset \theta_\alpha^\beta(\tau_\beta)$ , where  $\beta = \phi(\alpha)$ . Henceforth, we denote this continuous extension by the same symbol  $\rho_{\mu_\alpha}^l$ .

Take a net of bounded functions  $f_{\alpha,\kappa,y} \in L_{G_\beta}^1(G_\alpha)$  such that

$$(18) \quad \lim_\kappa \int_{G_\alpha} f_{\alpha,\kappa,y}(g) \mu_\alpha(dg) \mu_\beta(dh) = \delta_y(dh)$$

in  $M(G_\beta)$ , where  $\delta_y(dh)$  denotes the atomic Dirac measure on  $G_\beta$  with atom at  $y \in G_\beta$ ,  $\kappa \in K$ , where  $K$  is a directed set. Without loss of generality these functions can be chosen such that the linear span over the complex field  $\mathbf{C}$  of the family of functions  $\text{span}_{\mathbf{C}}\{f_{\alpha,\kappa,y} : y \in G_\beta, \kappa_0 < \kappa \in K\}$  is dense in  $L_{G_\beta}^1(G_\alpha)$  for each  $\kappa_0 \in K$ .

A subspace of continuous functions in  $L_{G_\beta}^1(G_\alpha)$  is dense in this space  $L_{G_\beta}^1(G_\alpha)$  (see also §10).

In the space  $L^1(G_\beta)$  the net  $\frac{\chi_{U_{\beta,w(s)}}}{\mu_\beta(U_{\beta,w(s)})}$  converges to the atomic Dirac measure  $\delta_{e_\beta}$  on  $G_\beta$ . Consider these family of functions related by the left shifts with weight factors

$$f_{\alpha,\kappa,y}(g) = \frac{f_{\alpha,\kappa,e_\alpha}(\theta_\alpha^\beta(y)g)}{\rho_{\mu_\alpha}^l(\theta_\alpha^\beta(y), e_\alpha)},$$

where  $y \in G_\beta$ , then we infer that

$$(19) \quad \int_{G_\alpha} f_{\alpha,\kappa,e_\alpha}(\theta_\alpha^\beta(hy)g)\mu_\alpha(dg)\mu_\beta(dh) = \int_{G_\alpha} f_{\alpha,\kappa,e}(s) \frac{\rho_{\mu_\alpha}^l(\theta_\alpha^\beta(hy), s)}{\rho_{\mu_\alpha}^l(\theta_\alpha^\beta(y), e_\alpha)} \mu_\alpha(ds)\mu_\beta(dh)$$

and

$$(20) \quad \lim_s \left( \frac{\chi_{U_{\beta,w}(s)}}{\mu_\beta(U_{\beta,w}(s))} \tilde{*} f_{\alpha,\kappa,e_\alpha}(s) \frac{\rho_{\mu_\alpha}^l(\theta_\alpha^\beta(hy), s)}{\rho_{\mu_\alpha}^l(\theta_\alpha^\beta(y), e_\alpha)} \right) = f_{\alpha,\kappa,e_\alpha}(g) \frac{\rho_{\mu_\alpha}^l(\theta_\alpha^\beta(y), g)}{\rho_{\mu_\alpha}^l(\theta_\alpha^\beta(y), e_\alpha)},$$

since  $e_\beta g = g$ .

Consider particularly  $f_{\alpha,v,e_\alpha} = \frac{\chi_{U_{\alpha,v}}}{\mu_\alpha(U_{\alpha,v})}$ . Applying Formulas (17 – 20) and 8(1) and Lemma 13 we deduce that the limit

$$\lim_s \lim_v \left( \frac{\chi_{U_{\beta,w}(s)}}{\mu_\beta(U_{\beta,w}(s))} \tilde{*} \frac{\xi_{\alpha,v}}{\mu_\alpha(U_{\alpha,v})} \right) = p_\alpha > 0$$

converges and is independent of  $h \in G_\beta$ , where  $\xi_{\alpha,v} := \|S_\alpha \chi_{U_{\alpha,v}}\|$ , since  $S$  is a bounded linear operator and

$$\lim_s \left\| \left( \frac{\chi_{U_{\beta,w}(s)}}{\mu_\beta(U_{\beta,w}(s))} \tilde{*} S_\alpha f \right) - S_\alpha f \right\|_{L_{G_\beta}^1(G_\alpha)} = 0$$

and

$$\lim_\kappa r \tilde{*} f_{\alpha,\kappa,y} = \int_{G_\beta} r(h) \delta_y(dh) = r(y)$$

for each continuous bounded function  $r$  on  $G_\beta$ . Therefore,

$$S_\alpha f_\alpha = p_\alpha \frac{\rho_{\mu_\alpha}^l(\theta_\alpha^\beta(a_\beta), e_\alpha)}{\rho_{\mu_\alpha}^l(\theta_\alpha^\beta(a_\beta), e_\alpha)} \hat{U}_{a_\beta} f_\alpha = p_\alpha \hat{U}_{a_\beta} f_\alpha$$

on  $L_{G_\beta}^1(G_\alpha)$  for each  $\alpha$ , where  $\beta = \phi(\alpha)$ . Moreover,

$$\|S\| = \sup_{\alpha \in \Lambda; \beta = \phi(\alpha)} p_\alpha \|\hat{U}_{a_\beta}\|_{L(X,X)}$$

with  $X = L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda$ , where  $L(X, Y)$  denotes the normed space of all bounded linear operators from  $X$  to  $Y$  with  $X$  and  $Y$  being complex normed spaces.

**14. Lemma.** *Let  $\hat{K}$  be the scalar continuous operator*

(1)  $\hat{K}f = pf$  for every  $f \in L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$ , that is  $K_\alpha f_\alpha = p_\alpha f_\alpha$  with  $p_\alpha > 0$  for each  $\alpha \in \Lambda$ . Then this operator  $\hat{K}$  satisfies Conditions 11(1–3) and

(2)  $f \sim \hat{K}f$ , that is  $f \sim u$  by the definition means that for every  $t$  if  $f_\alpha \geq 0$  and  $u_\alpha \geq 0$  and  $t_\alpha \geq 0$  for each  $\alpha \in \Lambda$  then  $\{f_\alpha \wedge t_\alpha = 0\} \Leftrightarrow \{u_\alpha \wedge t_\alpha = 0\}$  for any  $\alpha \in \Lambda$ , where  $f, u, t \in L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$ .

**Proof.** Properties 11(1–3) are evidently satisfied for  $\hat{K}$ . Condition (2) is also fulfilled, since  $\text{supp}(f_\alpha) = \text{supp}(p_\alpha f_\alpha)$  for each  $\alpha \in \Lambda$ .

**15. Theorem.** Topological group rings  $L^\infty(L_{G_\beta}^1(G_\alpha, \mu_\alpha) : \alpha < \beta \in \Lambda)$  and  $L^\infty(L_{G_\beta}^1(G_\alpha, \nu_\alpha) : \alpha < \beta \in \Lambda)$  are isomorphic if and only if measures  $\mu_\alpha$  and  $\nu_\alpha$  are equivalent for each  $\alpha \in \Lambda$ .

**Proof.** If measures are equivalent, an isomorphism of topological group rings (and algebras) is given by

$$(1) \quad L_{G_\beta}^1(G_\alpha, \mu_\alpha) \ni f_\alpha \mapsto f_\alpha \frac{d\mu_\alpha}{d\nu_\alpha} \in L_{G_\beta}^1(G_\alpha, \nu_\alpha) \quad \forall \alpha \in \Lambda.$$

Vice versa if topological group rings are isomorphic, all their representations in  $L(X, X)$  are equivalent, where  $X$  is a complex Banach space,  $L(X, X)$  denotes the Banach space of all continuous linear operators on  $X$  into  $X$ . Particularly, ring representations induced by unitary regular representations of groups  $G_\beta$  are also equivalent. A regular unitary representation  $T^{\beta, \mu_\alpha} : G_\beta \rightarrow U(X_\alpha)$  is prescribed by the formula

$$(2) \quad T^{\beta, \mu_\alpha}(h)f_\alpha(x) = \sqrt{\rho_{\mu_\alpha}(\theta_\alpha^\beta(h), x)} f_\alpha(\theta_\alpha^\beta(h^{-1})x),$$

where  $\beta = \phi(\alpha)$ ,  $\alpha \in \Lambda$ ,  $h \in G_\beta$ ,  $X_\alpha := L^2(G_\alpha, \mu_\alpha, \mathbf{C})$ ,  $U(X_\alpha)$  denotes the unitary group on the Hilbert space  $X_\alpha$ . The representation  $T^{\beta, \mu_\alpha}$  is strongly continuous on each  $G_\beta$  (see also [12] and References 55 and 181 and 195 there and §10 above).

The family of all simple functions of the form

$$(3) \quad f_\alpha = \sum_{j=1}^n b_j \chi_{B_{\alpha,j}}$$

with  $n \in \mathbf{N}$ ,  $b_j \in \mathbf{C}$ , open subsets  $B_{\alpha,j}$  in  $(G_\alpha, \tau_\alpha)$ ,  $B_{\alpha,j} \cap B_{\alpha,k} = \emptyset$  for all  $j \neq k$ , is dense in  $X_\alpha$ , since the measure  $\mu_\alpha$  is Borel regular. On the other

hand,  $B_{\alpha,j} \cap \theta_\alpha^\beta(G_\beta)$  are open in  $G_\beta$ , since  $\tau_\alpha \cap \theta_\alpha^\beta(G_\beta) \subset \theta_\alpha^\beta(\tau_\beta)$ . From the latter property it follows that

$$f_\alpha \circ \theta_\alpha^\beta = \sum_{j=1}^n b_j \chi_{K_{\beta,j}},$$

where  $K_{\beta,j}$  is open in  $G_\beta$  for each  $j$ . Therefore, the topological density  $d(X_\alpha)$  of the Hilbert space  $X_\alpha$  is not greater than that of  $X_\beta$ , consequently, there exists an isometric linear embedding

$$(4) \quad \eta_\beta^\alpha : X_\alpha \rightarrow X_\beta.$$

Let  $Y$  be a Banach space consisting of all vectors  $y = (y_\alpha : y_\alpha \in X_\alpha, \alpha \in \Lambda)$  with

$$\|y\| = \sup_{\alpha \in \Lambda} \|y_\alpha\|_{X_\alpha} < \infty,$$

it can be also denoted by  $Y = l_\infty(X_\alpha : \alpha \in \Lambda)$ . The embedding  $X_\alpha \oplus X_\beta \hookrightarrow Y$  induces the embedding  $L(X_\alpha, X_\alpha) \oplus L(X_\beta, X_\beta) \hookrightarrow L(Y, Y)$ . If a strongly continuous unitary representation  $T^\alpha : G_\alpha \rightarrow L(X_\alpha, X_\alpha)$  is given for each  $\alpha \in \Lambda$ , then taking Bochner's integral

$$R^\alpha(f_\alpha)y_\alpha := \int_{G_\alpha} f_\alpha(g)T^\alpha(g)y_\alpha\mu_\alpha(dg),$$

where  $f_\alpha \in L^1_{G_\beta}(G_\alpha)$  with  $\beta = \phi(\alpha)$  and  $y_\alpha \in X_\alpha$ , we get a strongly continuous representation  $R : L^\infty(L^1_{G_\beta}(G_\alpha, \mu_\alpha) : \alpha < \beta \in \Lambda) \rightarrow L(Y, Y)$ , since  $X_\alpha \oplus X_\beta$  has an embedding into  $Y$  and

$$\begin{aligned} R(u_\beta \tilde{*} f_\alpha)(y_\beta \oplus y_\alpha) &:= R\left[\int_{G_\beta} u_\beta(h)f_\alpha(\theta_\alpha^\beta(h)g)\mu_\beta(dh)\right]y_\beta \oplus y_\alpha = \\ &\int_{G_\beta} \int_{G_\alpha} (T^\beta(h)u_\beta(h)y_\beta) \oplus (f_\alpha(\theta_\alpha^\beta(h)g)T^\alpha(\theta_\alpha^\beta(h)g)y_\alpha\mu_\beta(dh)\mu_\alpha(dg) \\ &= R^\beta(u_\beta) \tilde{*} R^\alpha(f_\alpha)(y_\beta \oplus y_\alpha) \end{aligned}$$

for each  $\alpha \in \Lambda$  with  $\beta = \phi(\alpha)$ . Therefore,  $R(u \tilde{*} f) = R(u) \tilde{*} R(f)$  for each  $f, u \in L^\infty(L^1_{G_\beta}(G_\alpha, \mu_\alpha) : \alpha < \beta \in \Lambda)$ .

On the other hand, the embedding  $\eta_\beta^\alpha : X_\alpha \rightarrow X_\beta$  provides an equivalence relation  $\Sigma^\alpha$  so that  $T^{\beta, \mu_\alpha} \times T^{\gamma, \mu_\beta}$  induce a unitary representation  $T^{\beta, \gamma} : G_\beta \times G_\gamma \rightarrow U(X_\beta)$  for which

$$(5) \quad T^{\beta, \gamma}(g_\beta, g_\gamma) = \hat{T}^{\beta, \mu_\alpha}(g_\beta)T^{\gamma, \mu_\beta}(g_\gamma) \in U(X_\beta),$$

where a representation  $\hat{T}^{\beta, \mu_\alpha}$  is induced by  $T^{\beta, \mu_\alpha}$  and the embedding  $\eta_\beta^\alpha$ . Thus using an approximation of Dirac's measure  $\delta_z$  on  $G_\beta$  we get that an equivalence of two representations of two rings  $L^\infty(L_{G_\beta}^1(G_\alpha, \mu_\alpha) : \alpha < \beta \in \Lambda)$  and  $L^\infty(L_{G_\beta}^1(G_\alpha, \nu_\alpha) : \alpha < \beta \in \Lambda)$  provides an intertwining operator  $A^\beta$  of two regular unitary representations  $T^{\beta, \mu_\alpha}$  and  $T^{\beta, \nu_\alpha}$  such that  $A^\beta : L^2(G_\alpha, \nu_\alpha, \mathbf{C}) \rightarrow L^2(G_\alpha, \mu_\alpha, \mathbf{C})$  is a linear isomorphism of Hilbert spaces for each  $\alpha \in \Lambda$ . Measures  $\mu_\alpha$  and  $\nu_\alpha$  are regular Borel measures, while each group  $G_\alpha$  is Hausdorff, hence these regular representations distinguish different elements of  $G_\beta$ , that is they are injective. Thus

$$(6) \quad T^{\beta, \mu_\alpha} = A^\beta T^{\beta, \nu_\alpha} (A^\beta)^{-1}.$$

From this it follows, as it is known from the literature, that measures  $\mu_\alpha$  and  $\nu_\alpha$  are equivalent (see also [12] and References 55 and 181 and 195 there). We shortly recall a way of the proof.

A Borel subset  $J \in \mathcal{B}(G_\alpha)$  is of  $\mu_\alpha$  measure zero, i.e.  $\int_{G_\alpha} \chi_J d\mu_\alpha = 0$ , if and only if  $\int_{G_\alpha} k \chi_J d\mu_\alpha = 0$  for each nonnegative continuous function  $k$  on  $G_\alpha$ . If  $V$  is a linear topological isomorphism of  $L^2(G_\alpha, \mu_\alpha, \mathbf{C})$  onto itself, then  $\int_{G_\alpha} V(\chi_J) d\mu_\alpha = 0$ . Therefore, an operator  $V$  preserves invariant a set of nonnegative functions  $s$  on  $G_\alpha$  with  $\mu_\alpha(s) := \int_{G_\alpha} s d\mu_\alpha = 0$ , that is the family of all subsets in  $G_\alpha$  of  $\mu_\alpha$ -measure zero is invariant under  $V$ .

Consider matrix elements

$$(8) \quad (T^{\beta, \mu_\alpha}(g)x, y) = (A^\beta T^{\beta, \nu_\alpha}(g)(A^\beta)^{-1}x, y)$$

for each  $x, y \in X_\alpha$ . Evidently,  $\|y\| = 0$  if and only if the scalar product in (8) is zero for each  $x \in X_\alpha$  and  $g \in G_\beta$ . Sets  $E_{n, \mu_\alpha}$  and  $E_{n, \nu_\alpha}$  for measures  $\mu_\alpha$  and  $\nu_\alpha$  can be considered as  $E_n$  in §10. Then the limit

$$\lim_n \int_{G_\alpha} |(A^\beta)^{-1}[1 - \chi_{E_{n, \mu_\alpha}}]| d\nu_\alpha = 0$$

exists, since  $\lim_n \mu_\alpha(G_\alpha \setminus E_{n, \mu_\alpha}) = 0$  and the linear operator  $(A^\beta)^{-1}$  is continuous. Symmetrically

$$\lim_n \int_{G_\alpha} |A^\beta[1 - \chi_{E_{n, \nu_\alpha}}]| d\mu_\alpha = 0.$$

Each left shift  $L_h : G_\alpha \rightarrow G_\alpha$  with  $h \in \theta_\alpha^\beta(G_\beta)$ ,  $L_h g = hg$ , induces an isometry  $f_\alpha(g) \mapsto f_\alpha(hg)$  of  $L_{G_\beta}^1(G_\alpha, \mu_\alpha)$  onto itself and also for  $L_{G_\beta}^1(G_\alpha, \nu_\alpha)$ .



If  $\{U_{\alpha,v} : v\}$  is a base of neighborhoods of the unit element in  $G_\alpha$ , an arbitrary element  $q \in G_\alpha$  is marked, then there are elements  $g_{\beta,v,q} \in \theta_\alpha^\beta(G_\beta)$  such that  $g_{\beta,v,q}U_{\alpha,v}$  is a base of neighborhoods of  $q$ .

By Cauchy-Bounyakovskii's inequality  $|(T^{\beta,\mu_\alpha}(g)x_m, y)| \leq \|x_m\|\|y\|$ , since  $\|T^{\beta,\mu_\alpha}(g)\| = 1$ , hence if  $x_m \rightarrow 0$ , then  $(T^{\beta,\mu_\alpha}(g)x_m, y) \rightarrow 0$  uniformly by  $g \in G_\alpha$  with  $m$  tending to the infinity. If  $z_m \in L^2(G_\alpha, \mu_\alpha, \mathbf{C})$  are nonzero vectors, then  $b_m z_m / \|z_m\| \rightarrow 0$  for each sequence of complex numbers  $b_m$  tending to zero, when  $m$  tends to the infinity. A nonequivalence of measures would lead to a contradiction when one regular representation would be strongly continuous and another not on certain vectors, but these representations are equivalent and related by Formula (8). In view of Lemma 12 and Formulas (2, 8) for each Borel subset  $B$  in  $G_\alpha$ :  $\mu_\alpha(B) = 0$  if and only if  $\nu_\alpha(B) = 0$ , consequently, measures  $\mu_\alpha$  and  $\nu_\alpha$  are equivalent, since measures  $\mu_\alpha$  and  $\nu_\alpha$  are Borel regular.

**16. Theorem.** *Let  $G = \prod_{\alpha \in \Lambda} G_\alpha$  and  $H = \prod_{\alpha \in \Lambda} H_\alpha$  be two topological groups supplied with box topologies  $\tau_G^b$  and  $\tau_H^b$  respectively, where topological groups  $G_\alpha$  and  $H_\alpha$  for each  $\alpha \in \Lambda$  satisfy Conditions 1(1–4), measures  $\mu_\alpha$  on  $G_\alpha$  and  $\nu_\alpha$  on  $H_\alpha$  satisfy Conditions 2(1–4), a directed set  $\Lambda$  has not a minimal element.*

1. *If topological groups  $G_\alpha$  and  $H_\alpha$  for each  $\alpha \in \Lambda$  are topologically isomorphic, then equivalent measures  $\mu_\alpha$  and  $\nu_\alpha$  exist so that topological algebras  $L^\infty(L_{G_\beta}^1(G_\alpha, \mu_\alpha) : \alpha < \beta \in \Lambda)$  and  $L^\infty(L_{H_\beta}^1(H_\alpha, \nu_\alpha) : \alpha < \beta \in \Lambda)$  are isomorphic and their isomorphism  $\hat{T}$  satisfies properties (1–3) below.*

2. *If a bijective surjective continuous mapping  $\hat{T}$  of  $L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$  onto  $L^\infty(L_{H_\beta}^1(H_\alpha) : \alpha < \beta \in \Lambda)$  exists and  $\hat{T}^{-1}$  is continuous such that*

(1) *a mapping  $\hat{T} = (\hat{T}_\alpha f_\alpha : \alpha \in \Lambda)$  is linear so that  $\hat{T}_\alpha : L_{G_\beta}^1(G_\alpha) \rightarrow L_{H_\beta}^1(H_\alpha)$  for every  $\alpha \in \Lambda$  with  $\beta = \phi(\alpha)$ ;*

(2)  *$\hat{T}$  is positive, that is  $f_\alpha \geq 0$  in  $L_{G_\beta}^1(G_\alpha)$  if and only if  $\hat{T}_\alpha f_\alpha \geq 0$  in  $L_{H_\beta}^1(H_\alpha)$ ;*

(3)  *$\hat{T}$  is a ring homomorphism, that is  $\hat{T}(f \tilde{*} u) = (f \tilde{*} \hat{T}u)$  for each  $f, u \in L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$ ,*

*then topological groups  $G_\alpha$  and  $H_\alpha$  are topologically isomorphic and measures  $\mu_\alpha$  and  $\nu_\alpha$  are equivalent for each  $\alpha \in \Lambda$ .*

**Proof.** If  $\omega_\alpha : G_\alpha \rightarrow H_\alpha$  is a topological group isomorphism for each

$\alpha \in \Lambda$ , then an operator  $\hat{T}$  with  $(\hat{T}_\alpha f_\alpha)(h) := f_\alpha(\omega_\alpha^{-1}(h))$  for every  $\alpha \in \Lambda$  and  $h \in H_\alpha$  and  $f_\alpha \in L_{G_\beta}^1(G_\alpha)$  has the desired properties. Taking a measure  $\nu_\alpha = \mu_\alpha \circ \omega_\alpha^{-1}$  on  $H_\alpha$  for any  $\alpha \in \Lambda$  establishes an isometric isomorphism  $\hat{T}$  which satisfies Conditions (1 – 3).

Conversely, let  $\hat{T}$  satisfy the conditions of this theorem. Then from the conditions of this theorem the algebraic isomorphisms

(4)  $\hat{U}(G) \cong \hat{U}(H)$  and  $\hat{S}(G) \cong \hat{S}(H)$  follow, where  $\hat{U}(G)$  and  $\hat{S}(G)$  denote the families of all operators satisfying Conditions 11(1–3) and (11(1–3), 14(1, 2)) correspondingly. These algebraic homomorphisms are induced by the operator  $\hat{T}$  according to the formula

$$(5) \quad \hat{U}(G) \ni \hat{P} \mapsto \hat{T}\hat{P}\hat{T}^{-1} \in \hat{U}(H).$$

In view of Theorem 11 and Lemma 14 there are algebraic isomorphisms  $\hat{G} \cong \hat{U}(G)/\hat{S}(G)$  and  $\hat{H} \cong \hat{U}(H)/\hat{S}(H)$ , since the group algebra  $L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$  is isomorphic with  $L^\infty(L_{H_\beta}^1(H_\alpha) : \alpha < \beta \in \Lambda)$ , where  $\hat{G}$  and  $\hat{H}$  denote the groups of left translation operators from §11. Therefore, algebraically the group  $\hat{G}$  is isomorphic with  $G$  and the group  $\hat{H}$  with  $H$  respectively, since a directed set  $\Lambda$  has not a minimal element. But the mappings  $\hat{T}$  and  $\hat{T}^{-1}$  are bijective, surjective and continuous, applying (3) and 11(3, 17, 18, 20) we get the topological isomorphism  $\omega_\alpha$  of  $G_\alpha$  with  $H_\alpha$  for each  $\alpha \in \Lambda$ .

In view of Theorem 15 measures  $\mu_\alpha \circ \omega_\alpha^{-1}$  and  $\nu_\alpha$  on  $H_\alpha$  are equivalent and the isometric isomorphism of group algebras is provided by the mapping

$$L_{G_\beta}^1(G_\alpha, \mu_\alpha) \ni f_\alpha \mapsto (f_\alpha \circ \omega_\alpha^{-1}) \frac{d\mu_\alpha \circ \omega_\alpha^{-1}}{d\nu_\alpha} \in L_{H_\beta}^1(H_\alpha, \nu_\alpha) \quad \forall \alpha \in \Lambda.$$

**17. Definition.** Let

$$(1) \quad (g_\beta * f_\alpha)(s) = \int_{G_\beta} g_\beta(x) f_\alpha(\theta_\alpha^\beta(x^{-1})s) \mu_\beta(dx) \text{ and}$$

$$(2) \quad (g_\beta \check{*} f_\alpha)(s) = \int_{G_\beta} g_\beta(x) f_\alpha(s\theta_\alpha^\beta(x^{-1})) \mu_\beta(dx)$$

for each  $s \in G_\alpha$ .

The group algebra  $L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$  will be called meta-commutative, if the following condition is satisfied:

$$(3) \quad \left[ \frac{f_\beta(x)}{\psi_\beta(x^{-1}\theta_\beta^\gamma(s))\rho_{\mu_\beta}^l(\theta_\beta^\gamma(s^{-1}), \theta_\beta^\gamma(s^{-1})x)} * g_\alpha(\theta_\alpha^\beta(x^{-1})\theta_\alpha^\gamma(s)) \right] (\theta_\alpha^\gamma(s))$$

$$= [g_\beta(y) \check{*} f_\alpha(\theta_\alpha^\gamma(s) \theta_\alpha^\beta(y^{-1}))](\theta_\alpha^\gamma(s))$$

for each  $\alpha \in \Lambda$ , every  $s \in G_\gamma$  with  $\beta = \phi(\alpha)$ ,  $\gamma = \phi(\beta)$ , for every  $f, g \in L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$  such that  $g_\alpha \circ \theta_\alpha^\beta = g_\beta$  and

$$(4) f_\alpha \circ \theta_\alpha^\beta = f_\beta \text{ on } G_\beta.$$

**18. Theorem.** *The group algebra  $L^\infty(L_{G_\beta}^1(G_\alpha) : \alpha < \beta \in \Lambda)$  is meta-commutative if and only if a group  $G$  is commutative.*

**Proof.** A group  $G$  is commutative if and only if a group  $G_\alpha$  is commutative for each  $\alpha \in \Lambda$ . Since there are approximations of Dirac's measure  $\delta_z$  on  $G_\alpha$  in this group algebra, then the group  $G_\alpha$  is commutative if and only if

$$(1) f_\alpha(y^{-1}s) = f_\alpha(sy^{-1}) \text{ for each } f_\alpha \in L_{G_\beta}^1(G_\alpha) \text{ and any } x, s \in G_\alpha.$$

On the other hand, each continuous bounded function  $f_\alpha$  on  $G_\alpha$  is also continuous on  $\theta_\alpha^\beta(G_\beta)$  relative to the topology  $\theta_\alpha^\beta(\tau_\beta)$ , since  $\tau_\alpha \cap \theta_\alpha^\beta(G_\beta) \subset \theta_\alpha^\beta(\tau_\beta)$ , consequently,  $f_\alpha$  has a continuous bounded restriction  $f_\beta = f_\alpha \circ \theta_\alpha^\beta$  on the topological space  $(G_\beta, \tau_\beta)$ . This restriction  $f_\beta$  satisfies Condition 17(4). The space of bounded continuous functions  $f_\alpha$  satisfying 17(4) is dense in  $L_{G_\beta}^1(G_\alpha)$ .

Therefore, it is sufficient to demonstrate that Equality (1) is equivalent to 17(3) for any  $y, s \in \theta_\alpha^\gamma(G_\gamma)$ , since  $G_\gamma$  is dense in  $G_\alpha$  and in  $G_\beta$ , where  $\beta = \phi(\alpha)$  and  $\gamma = \phi(\beta)$ . From Formulas 17(1, 2) we infer that

$$\begin{aligned} (2) \quad & \left[ \frac{f_\beta(x)}{\psi_\beta(x^{-1}\theta_\beta^\gamma(s))\rho_{\mu_\beta}^l(\theta_\beta^\gamma(s^{-1}), \theta_\beta^\gamma(s^{-1})x)} * g_\alpha(\theta_\alpha^\beta(x^{-1})\theta_\beta^\gamma(s)) \right](\theta_\beta^\gamma(s)) = \\ & \int_{G_\beta} \frac{f_\beta(x)}{\psi_\beta(x^{-1}\theta_\beta^\gamma(s))\rho_{\mu_\beta}^l(\theta_\beta^\gamma(s^{-1}), \theta_\beta^\gamma(s^{-1})x)} g_\alpha(\theta_\alpha^\beta(x^{-1})\theta_\beta^\gamma(s)) \mu_\beta(dx) \\ & = \int_{G_\beta} g_\beta(y) f_\alpha(\theta_\alpha^\gamma(s) \theta_\alpha^\beta(y^{-1})) \frac{\psi_\beta(y) \rho_{\mu_\beta}^l(\theta_\beta^\gamma(s^{-1}), y^{-1})}{\psi_\beta(y) \rho_{\mu_\beta}^l(\theta_\beta^\gamma(s^{-1}), y^{-1})} \mu_\beta(dy) \end{aligned}$$

after the change of the integration variable  $y = x^{-1}\theta_\beta^\gamma(s)$ , where  $x \in G_\beta$  and  $s \in G_\gamma$ . Therefore, from Formulas (2) and 17(1, 2) it follows, that Condition 17(3) is equivalent to the equality

$$(3) \quad \int_{G_\beta} g_\beta(x) f_\alpha(\theta_\alpha^\beta(x^{-1})\theta_\beta^\gamma(s)) \mu_\beta(dx) = \int_{G_\beta} g_\beta(x) f_\alpha(\theta_\beta^\gamma(s) \theta_\alpha^\beta(x^{-1})) \mu_\beta(dx).$$

A bounded continuous function  $g_\beta$  is arbitrary in  $L_{G_\gamma}^1(G_\beta)$  in the latter formula, consequently, Equality (1) is satisfied if and only if 17(3) is valid.

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